Lecture Notes on **Discrete Mathematics.**
Birzeit University, Palestine, 2020

Number Theory
and Proof Methods

Mustafa Jarrar

4.1 Introduction
4.2 Rational Numbers
4.3 Divisibility
4.4 Quotient-Remainder Theorem

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Acknowledgement:
This lecture is based on (but not limited to) to chapter 4 in "Discrete Mathematics with Applications by Susanna S. Epp (3rd Edition)".

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Number Theory

4.4 Quotient-Remainder Theorem

In this lecture:

- Part 1: **Quotient-Remainder Theorem**
- Part 2: *div* and *mod*, and applications in real-life
- Part 3: Representing Integers in Quotient-Remainder
- Part 4: Absolute Value

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Quotient-Remainder Theorem

Notice that: $4\overline{)\,11}$ with quotient 2 and remainder 8, 3

$11 = 4\times2 + 3$

$d\overline{)\,n}$ with quotient q and remainder r

Theorem 4.4.1 The Quotient-Remainder Theorem

Given any integer n and positive integer d, there exist unique integers q and r such that

$$n = dq + r \quad \text{and} \quad 0 \le r < d$$

Examples:

$54 = 4 \cdot 13 + 2$ $\quad q = 13$ $\quad r = 2$

$-54 = 4 \cdot (-14) + 2$ $\quad q = -14$ $\quad r = 2$

$54 = 70 \cdot 0 + 54$ $\quad q = 0$ $\quad r = 54$

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4.4 Quotient-Remainder Theorem

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- Part 1: Quotient-Remainder Theorem
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div and mod

Definition

Given an integer n and a positive integer d,

n div d = the integer quotient obtained when n is divided by d, and

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n mod d = the nonnegative integer remainder obtained when n is divided by d.

"%" in C,JAVA
"\" in .net

Symbolically, if n and d are integers and $d > 0$, then

$$n \text{ div } d = q \text{ and } n \bmod d = r \iff n = dq + r$$

Where q and r are integers and $0 \le r < d$

Examples:

$$32 \text{ div } 9 = 3$$
$$32 \bmod 9 = 5$$

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Application of div and mod

Computing the Day of the Week

Suppose today is Tuesday, and neither this year nor next year is a leap year (ليست سنة كبيسة). What day of the week will it be 1 year from today?

365 div 7 = 52 *and* *365 mod 7 = 1*

It means that a year (365) is 52 weeks + 1 day

So,

after 364 it will be Tuesday

*and after 365 it will be **Wednesday***

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Application of div and mod

Computing the Day of the Week

If today is Thursday and it is 16/10/2014, which day it will be the valentine's day in 2015?

Valentine's day = 14/2/2015

The number of days from today to 14/2/2015 =

15 in October
+ **30** in November
+ **31** in December
+ **31** in January
+ **14** in February
= 121 days

So?

121 div 7 = 17

121 mod 7 = 2

That is, after 17 weeks the day will be Thursday, and two days after, it will be: Saturday

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Application of div and mod

Solving a Problem about *mod*

Suppose *m* is an integer. If *m mod* 11 = 6, what is 4*m mod* 11?

$m = 11q + 6$

So,

$4m = 44q + 24$

$= 44q + 22 + 2$

$= 11(4q + 2) + 2$ $(4q + 2)$ is integer

Thus, $4m \ mod \ 11 = 2$

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4.4 Quotient-Remainder Theorem

In this lecture:

- ❑ Part 1: Quotient-Remainder Theorem
- ❑ Part 2: *div* and *mod*, and applications in real-life
- ❑ Part 3: **Representing Integers in Quotient-Remainder**
- ❑ Part 4: Absolute Value

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Representing Integers using the quotient-remainder theorem

Parity Property

We represent any number as:

$$n = 2q + r \quad \text{and} \quad 0 \le r < 2$$

Because we have only r = 0 and r =1, then:

$$n = 2q + 0 \quad \text{or} \quad n = 2q + 1$$

Even Odd

Therefore, n is either even or odd (parity)

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Representing Integers using the quotient-remainder theorem

Proving Parity Property

Theorem 4.4.2 The Parity of Property

Any two consecutive integers have opposite parity

Given m and $m+1$ are consecutive integers.

Then, one is odd and the other is even (by parity property)

Case 1 (m is even): $m = 2k$

$So, m+1 = 2k+1$, which is odd

Case 2 (m is odd): $m = 2k + 1$

$So, m+1 = (2k+1) + 1$

$= 2k + 2$

$= 2(k+1)$. thus $m + 1$ is even.

Proof by **division into cases**

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The "divide into cases" Proof Method

Method of Proof by Division into Cases

To prove a statement of the form "If A_1 or A_2 or ... or A_n, then C," prove all of the following:

If A_1, then C,
If A_2, then C,
.
.
.
If A_n, then C.

This process shows that C is true regardless of which of $A_1, A_2, ..., A_n$, happens to be the case.

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Representing Integers using the quotient-remainder theorem

Integers Modulo 4

We represent any integer as:

$$n=4q \quad \text{or} \quad n=4q+1 \quad \text{or} \quad n=4q+2 \quad \text{or} \quad n=4q+3$$

This implies that there exist an integer quotient q and a remainder r such that

$$n = 4q + r \quad \text{and} \quad 0 \le r < 4.$$

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Using the "divide into cases" Method

Theorem 4.4.3

The square of any odd integer has the form $8m + 1$ for some integer m.

Hint: any odd integer can be (4q+1) or (4q+3).

$$\forall n \in Z^{odd}, \quad \exists m \in Z \ . \ n^2 = 8m + 1$$

Case 1 (n=4q+1): $n^2 = 8m + 1 = (4q+1)^2$

$= 16q^2 + 8q + 1$

$= 8(2q^2 + q) + 1$ — $(2q^2 + q)$ is an integer m

thus $n^2 = 8m + 1$

Case 2 (n=4q+3): $n^2 = 8m + 1 = (4q+3)^2$

$= 16q^2 + 24q + 8 + 1$

$= 8(2q^2 + 3q+1) + 1$ — $(2q2 + 3q+1)$ is an integer m

thus $n^2 = 8m + 1$

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Congruence Modulo

$A \equiv B \pmod{C}$ — A is **congruent** to B modulo C. → $C \mid (A-B)$

1. ≡ is the symbol for congruence (!توافق), which means the values A and B are in the same **equivalence class**.
2. (mod C) tells us what **operation** we applied to ***A*** and ***B***.
3. when we have both of these, we call "≡" **congruence modulo C.**

e.g. $26 \equiv 11 \pmod{5}$ → $5 \mid (26-11)$

e.g. $a \equiv 3 \pmod{2}$ → $2 \mid (a-3)$

Given: $x \equiv -2 \pmod{2}$

Which of the following integers are valid solutions for x ?

✗ -49 ✓ -44

✓ 26 ✗ -23

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- Part 1: Quotient-Remainder Theorem
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- Part 3: Representing Integers in Quotient-Remainder
- Part 4: **Absolute Value**

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Absolute Value

القيمة المطلقة

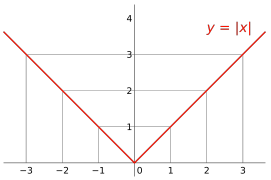
Definition

For any real number x, the **absolute value of** x, denoted $|x|$, is defined as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example:

$|2| = 2$
$|-2| = 2$



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Try it yourself

Absolute Value

Lemma 4.4.4

For all real numbers r, $-|r| \le r \le |r|$

Proof: Suppose r is any real number.

Case 1 ($r \ge 0$):

$|r| = r$ by definition,

$-|r| < r$ as r is positive and so $-|r|$ is negative

$\therefore \quad -|r| \le r \le |r|$

Case 2 ($r < 0$):

$|r| = -r$ by definition,

thus, $-|r| = r$

$r < |r|$ as r is negative and $|r|$ is positive

$\therefore \quad -|r| \le r \le |r|$

Thus, in either case, $-|r| \le r \le |r|$

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Try it yourself

Absolute Value

Lemma 4.4.5

For all real numbers r, $|-r| = |r|$

Suppose r is any real number. By Theorem T23 in Appendix A, if $r > 0$, then $-r < 0$, and if $r < 0$, then $-r > 0$. Thus

$$|-r| = \begin{cases} -r & \text{if } -r > 0 \\ 0 & \text{if } -r = 0 \\ -(-r) & \text{if } -r < 0 \end{cases}$$ by definition of absolute value

$$= \begin{cases} -r & \text{if } -r > 0 \\ 0 & \text{if } -r = 0 \\ r & \text{if } -r < 0 \end{cases}$$ because $-(-r) = r$ by Theorem T4 in Appendix A

$$= \begin{cases} -r & \text{if } r < 0 \\ 0 & \text{if } -r = 0 \\ r & \text{if } r > 0 \end{cases}$$ because, by Theorem T24 in Appendix A, when $-r > 0$, then $r < 0$, when $-r < 0$, then $r > 0$, and when $-r = 0$, then $r = 0$

$$= \begin{cases} r & \text{if } r \ge 0 \\ -r & \text{if } r < 0 \end{cases}$$ by reformatting the previous result

$= |r|$ by definition of absolute value.

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Try it yourself

Absolute Value and Triangle Inequality

Theorem 4.4.6 The Triangle Inequality

For all real numbers x and y, $|x+y| \le |x| + |y|$

Case 1 ($x + y \ge 0$):

$|x + y| = x + y$ by Lemma 4.4.4

so $x \le |x|$ & $y \le |y|$

$\therefore$ $|x + y| = x + y \le |x| + |y|$

Case 2 ($x + y < 0$):

$|x + y| = -(x + y) = -x + -y$ by Lemmas 4.4.4 &4.4.5

so $-x \le |-x| = |x|$ and $-y \le |-y| = |y|$.

$\therefore$ $|x + y| = (-x) + (-y) \le |x| + |y|$.

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